Statistical Modeling & Path Analysis

Theory Construction and Statistical Modeling

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Outline

Flavors of Statistical Analysis

- Statistical Reasoning
- Statistical Testing
- Statistical Modeling

Path Analysis

Motivating Example

Imagine you are working for an F1 team. You’re job is to use data from past seasons to optimize the baseline setup of your team’s car.

- Suppose you have two candidate setups that you want to compare.
- For each setup, you have 100 past lap times.
- How do you distill those 200 lap times into a succinct decision between the two setups?

Motivating Example

Suppose I tell you that the mean lap time for Setup A is 118 seconds and the mean lap time for Setup B is 110 seconds.

- Can you confidently recommend Setup B?
- What caveats might you consider?

Motivating Example

Suppose I tell you that the standard deviation for the times under Setup A is 7 seconds and the standard deviation for the times under Setup B is 5 seconds.

- How would you incorporate this new information into your decision?

Motivating Example

Suppose I tell you that the standard deviation for the times under Setup A is 7 seconds and the standard deviation for the times under Setup B is 5 seconds.

- How would you incorporate this new information into your decision?

Suppose, instead, that the standard deviation of times under Setup A is 35 seconds and the standard deviation under setup B is 25 seconds.

- How should you adjust your appraisal of the setups’ relative benefits?

Statistical Reasoning

The preceding example calls for *statistical reasoning*.

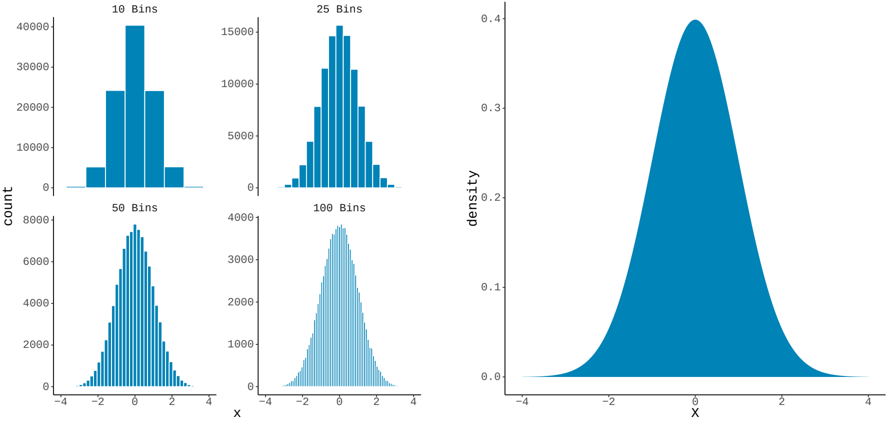
- The foundation of all good statistical analyses is a deliberate, careful, and thorough consideration of uncertainty.
- In the previous example, the mean lap time for Setup A is clearly longer than the mean lap time for Setup B.
- If the times are highly variable, with respect to the size of the mean difference, we may not care much about the mean difference.
- The purpose of statistics is to systematize the way that we account for uncertainty when making data-based decisions.

Probability Distributions

Statisticians (and anyone who uses statistics) quantify uncertainty using probability distributions.

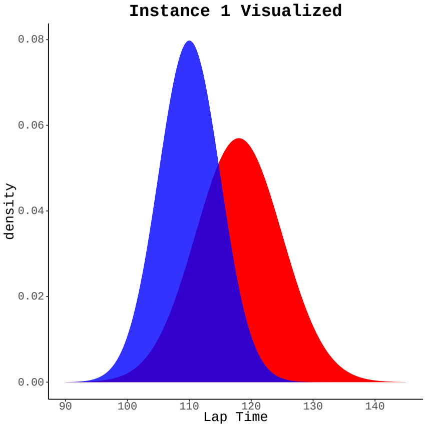
- Probability distributions quantify how likely it is to observe each possible value of some probabilistic entity.
- Probability distributions are re-scaled frequency distributions.
- We can build up the intuition of a probability density by beginning with a histogram.

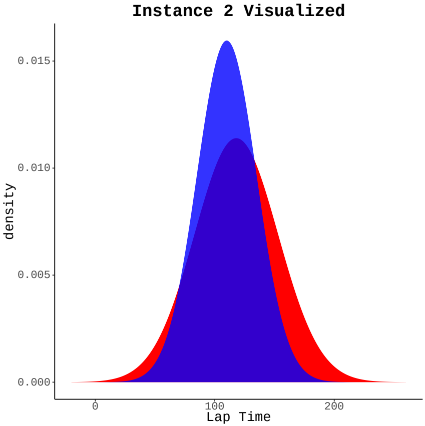
Probability Distributions



Reasoning with Distributions

We will gain insight by conceptualizing our example problem in terms of the underlying distributions of lap times.


Instance 1 Visualized


Instance 2 Visualized

Statistical Testing

In practice, we may want to distill the information in the preceding plots into a simple statistic so we can make a judgment.

- One way to distill this information and control for uncertainty when generating knowledge is through statistical testing.
  - When we conduct statistical tests, we define a *test statistic* by weighting the estimated effect by the precision of the estimate.
- One of the most common test statistics, *Student's t-test*, follows this pattern:

$$t = \frac{\textit{Estimate} - \textit{Null-Hypothesized Value}}{\textit{Variability}}$$

Statistical Testing

To test the nil-null hypothesis of a zero mean difference, we define the t-statistic as follows:

$$t = \frac{\left(\bar{X}_A - \bar{X}_B\right) - 0}{\sqrt{S^2_{A-B}\left(n_A^{-1} + n_B^{-1}\right)}}$$

where

$$Estimate = \bar{X}_A - \bar{X}_B$$

and

$$\begin{aligned} Variability &= \sqrt{S^2_{A-B}\left(n_A^{-1} + n_B^{-1}\right)} \\ &= \sqrt{\frac{(n_A - 1)S_A^2 + (n_B - 1)S_B^2}{n_A + n_B - 2}\left(\frac{1}{n_A} + \frac{1}{n_B}\right)} \end{aligned}$$

Statistical Testing

Applying the preceding formula to the first instantiation of our example problem produces:

$$t = \frac{118 - 110 - 0}{\sqrt{\frac{(100-1)7^2+(100-1)5^2}{100+100-2}\left(\frac{1}{100}+\frac{1}{100}\right)}}$$

$$\approx \frac{8}{0.86}$$

$$\approx 9.30$$

Statistical Testing

If we consider the second instantiation of our example problem, the effect does not change, but our measure of variability does:

$$V = \sqrt{\frac{(100-1)35^2 + (100-1)25^2}{100+100-2}\left(\frac{1}{100}+\frac{1}{100}\right)}$$
$$\approx 4.30$$

As a results, our test statistic changes to reflect our decreased certainty:

$$t \approx \frac{8}{4.30} \approx 1.86$$

Statistical Modeling

Statistical testing is a very useful tool, but it quickly reaches a limit.

- In experimental contexts (with successful random assignment) real-world “messiness” is controlled through random assignment.
    - Researchers working with messy observational data, usually don’t have questions that lend themselves to rigorous testing.
- Unless embedded in a larger model, statistical tests can only answer simple yes/no questions about single parameters.
    - Researchers studying complex processes need more elaborate means of representing the phenomena under study.

Such situations call for *statistical modeling*.

What is a Statistical Model?

A statistical model is a mathematical representation of the thing we’re trying to study.

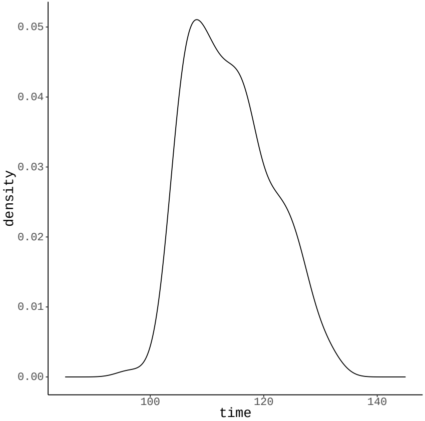
- We can basically model anything.
  - Theoretical process
  - Social or physical system
  - Natural phenomenon
- The model succinctly describes whatever system is being analyzed.
  - The model is an abstraction of reality.
  - We only include the *interesting* parts of the process.
- For our purposes, a statistical model is a probability distribution that describes the possible ways our focal system can ”behave”.
  - Such model are a rigorous, unambiguous quantification of a theory.
  - We can evaluate theories by comparing models.

Statistical Modeling

To apply a modeling approach to our example problem we consider the combined distribution of lap times.

- The model we construct will explain variation in lap times based on interesting features.

- In this simple case, the only feature we consider is the type of setup.



Modeling our Example

Let's say we're willing to assume that the (conditional) distribution of lap times is normal.

$$Y_{time} \sim \mathrm{N}\left(\mu, \sigma^2\right)$$

To get the same answer as our statistical test, we model the mean of the distribution of lap times, μ, using a single grouping factor.

$$\mu = \beta_0 + \beta_1 X_{setup}$$

$$Y_{time} \sim \mathrm{N}\left(\beta_0 + \beta_1 X_{setup}, \sigma^2\right)$$

Modeling our Example

Since we're mostly interested in describing the mean lap time, we can express the above differently:

$$Y_{time} = \beta_0 + \beta_1 X_{setup} + \varepsilon$$

$$\varepsilon \sim \mathrm{N}\left(0, \sigma^2\right)$$

After we fit this model to a sample, the parameters β_0 and β_1 are replaced by estimated statistics.

$$\hat{Y}_{time} = \hat{\beta}_0 + \hat{\beta}_1 X_{setup}$$

$$= 110 + 8X_{setup}$$

Modeling our Example

We can easily fit this model in R:

```
lmOut <- lm(time ~ setup, data = exData)

partSummary(lmOut, -c(1, 2))

## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 110.0000     0.6083   180.8   <2e-16
## setupA        8.0000     0.8602     9.3   <2e-16
##
## Residual standard error: 6.083 on 198 degrees of freedom
## Multiple R-squared:  0.304,Adjusted R-squared:  0.3005
## F-statistic: 86.49 on 1 and 198 DF,  p-value: < 2.2e-16
```

Two Modeling Traditions

Breiman (2001) defines two cultures of statistical modeling:

- Data models & Algorithmic models
- Our definition of *statistical models* matches Breiman's definition of *data models*.

Two Modeling Traditions

Breiman (2001) defines two cultures of statistical modeling:

- Data models & Algorithmic models
- Our definition of *statistical models* matches Breiman's definition of *data models*.

Both approaches have strengths and weaknesses.

- Data models tend to support a priori hypothesis testing more easily.
- Data models also tend to provide more interpretable results.
- Algorithmic models are currently preferred in cutting edge prediction/classification applications.
- Many models can be viewed as data models or algorithmic models, depending on how they're used.

Characteristics of Models

DATA MODELS

- Data models are built from probability distributions.
    - Data models are modular.
- Data models encode our hypothesized understanding of the system we're exploring.
    - Data models are constructed in a "top-down", theory-driven way.

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Algorithmic Models

- Algorithmic models do not have to be built from probability distributions.
  - Often, they are based on a set of decision rules (i.e., an algorithm).
- Algorithmic models begin with an objective (i.e., a problem to solve) and seek the optimal solution, given the data.
  - They are built in a "bottom-up", data-driven way.

Data Modeling Example

Suppose we believe the following:

1. BMI is positively associated with disease progression in diabetic patients after controlling for age and average blood pressure.
2. After controlling for age and average blood pressure, the effect of BMI on disease progression is different for men and women.

We can represent these beliefs with a moderated regression model:

$$Y_{prog} = \beta_0 + \beta_1 X_{BMI} + \beta_2 X_{sex} + \beta_3 X_{age} + \beta_4 X_{BP} + \beta_5 X_{BMI} X_{sex} + \varepsilon$$

Data Modeling Example

We can use R to fit our model to some patient data:

```
## Load the data:
dataDir  <- "../data/"
diabetes <- readRDS(paste0(dataDir, "diabetes.rds"))

## Fit the regression model:
fit <- lm(progress ~ bmi * sex + age + bp, data = diabetes)
```

Data Modeling Example

```
partSummary(fit, -c(1, 2))

## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -174.7986    27.0004  -6.474 2.58e-10
## bmi            7.2106     0.8922   8.082 6.34e-15
## sexmale      -90.1718    35.1134  -2.568   0.0106
## age            0.1691     0.2322   0.728   0.4670
## bp             1.4032     0.2385   5.884 7.97e-09
## bmi:sexmale    3.0257     1.3090   2.311   0.0213
##
## Residual standard error: 59.68 on 436 degrees of freedom
## Multiple R-squared:  0.4075,Adjusted R-squared:  0.4007
## F-statistic: 59.98 on 5 and 436 DF,  p-value: < 2.2e-16
```

Data Modeling Example

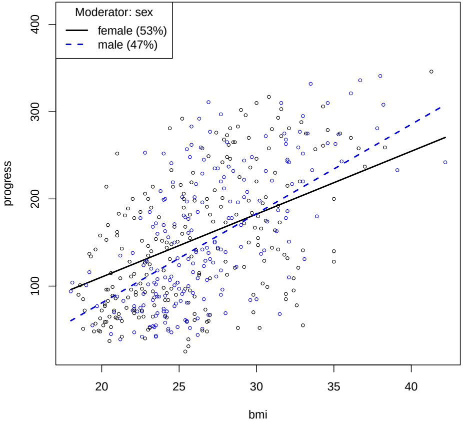
We can do a simple slopes analysis to test the group-specific effects of BMI on disease progression:

```
library(rockchalk)

psOut <- plotSlopes(fit, plotx = "bmi", modx = "sex")
tsOut <- testSlopes(psOut)
```

```
tsOut$hypotests[ , -1]

##            slope Std. Error  t value     Pr(>|t|)
## female  7.210575  0.8921929 8.081856 6.335264e-15
## male   10.236323  1.0328739 9.910525 5.137409e-21
```

Data Modeling Example

We can also visualize the simple slopes:



Algorithmic Modeling Example

Suppose we want to find the best predictors of disease progression among the variables contained in our dataset:

- Age
- BMI
- Blood Pressure
- Blood Glucose
- Sex
- Total Cholesterol
- LDL Cholesterol
- HDL Cholesterol
- Triglycerides
- Lamorigine

We could try *best-subset selection*.

- Fit a series of regression models wherein disease progression is predicted by all possible subsets of X variables.
- Choose the set of X variables that minimizes the prediction error.

Algorithmic Modeling Example

```r
library(leaps)

## Save the predictor variables' names:
xNames <- grep(pattern = "progress",
               x       = colnames(diabetes),
               invert  = TRUE,
               value   = TRUE)

## Train the models:
fit <- regsubsets(x     = progress ~ .,
                  data  = diabetes,
                  nvmax = ncol(diabetes) - 1)

## Summarize the results:
sum <- summary(fit)
```

Algorithmic Modeling Example

```
sum$outmat

##           age bmi bp  tc  ldl hdl tch ltg glu sexmale
## 1  ( 1 )  " " "*" " " " " " " " " " " " " " " " "
## 2  ( 1 )  " " "*" " " " " " " " " " " "*" " " " "
## 3  ( 1 )  " " "*" "*" " " " " " " " " "*" " " " "
## 4  ( 1 )  " " "*" "*" "*" " " " " " " "*" " " " "
## 5  ( 1 )  " " "*" "*" " " " " "*" " " "*" " " "*"
## 6  ( 1 )  " " "*" "*" "*" "*" " " " " "*" " " "*"
## 7  ( 1 )  " " "*" "*" "*" "*" " " "*" "*" " " "*"
## 8  ( 1 )  " " "*" "*" "*" "*" " " "*" "*" "*" "*"
## 9  ( 1 )  " " "*" "*" "*" "*" "*" "*" "*" "*" "*"
## 10  ( 1 ) "*" "*" "*" "*" "*" "*" "*" "*" "*" "*"
```

Algorithmic Modeling Example

```
## Variables selected by BIC:
xNames[with(sum, which[which.min(bic), -1])]

## [1] "bmi" "bp"  "hdl" "ltg" "sex"

## Variables selected by Adjusted R^2:
xNames[with(sum, which[which.max(adjr2), -1])]

## [1] "bmi" "bp"  "tc"  "ldl" "tch" "ltg" "glu" "sex"

## Variables selected by Mallow's Cp:
xNames[with(sum, which[which.min(cp), -1])]

## [1] "bmi" "bp"  "tc"  "ldl" "ltg" "sex"
```

Prediction & Estimation

There are two other common objectives of statistical analyses.

1. Prediction/Classification
2. Estimation

Prediction/Classification involves building a model to ”guess” future values of some outcome.

- Weather forecasting
- Predicting the winner of an election
- Financial projections

Estimation focuses on getting the most accurate possible estimate of some real-world quantity.

- The number of refugees in a country
- The rates of obesity in a certain population
- The number of traffic accidents in a given area

Formal Modeling

Smaldino (2017) distinguishes between two ways in which we can translate theories into models.

Verbal Model

- Vague description of the theory or phenomenon
- Does not describe the theory with enough rigor/specificity to define a single, unambiguous representation
- Could describe multiple phenomena equally well

Formal Model

- Rigorously defines all the important aspects of a system
- Implies only one representation of the phenomenon
- Can be used to rule-out potential theories by comparing models

PATH ANALYSIS

Path Analysis

Suppose we have the following theory about diabetic patients.

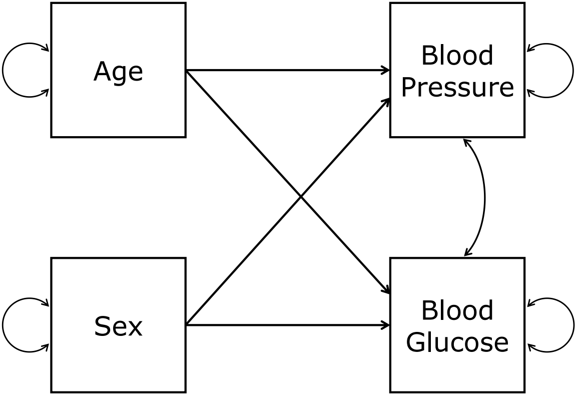
- A patient's age and sex affect their blood pressure and blood glucose levels.
- After accounting for age and sex, blood pressure and blood glucose levels retain some residual correlation.
- Age and sex are not correlated.

This theory implies two correlated outcome variables.

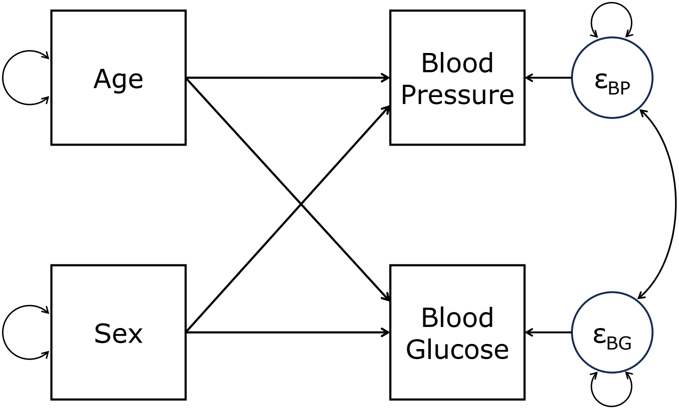
- We cannot model this theory with univariate regression models.

This is a prime case for *path analysis*.

Path Diagram



Path Diagram



Estimating the Model

```r
library(dplyr)
library(lavaan)

mod1 <- '
## Define the structural relations:
bp + glu ~ age + male

## Do not allow the input variables to covary:
age ~~ 0 * male
'

out <- diabetes %>%
  mutate(male = ifelse(sex == "male", 1, 0)) %>%
  sem(mod1, data = ., fixed.x = FALSE)
```

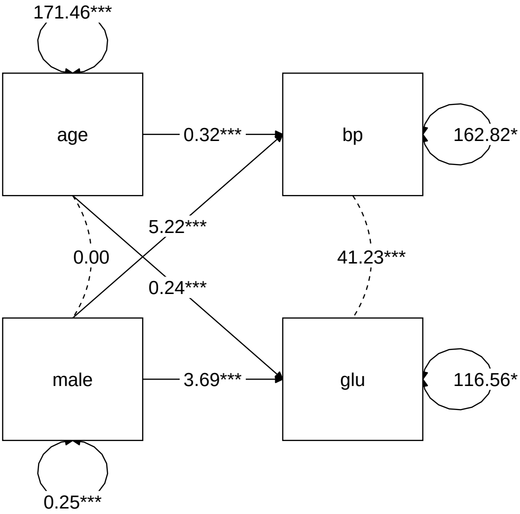
Estimating the Model

```
partSummary(out, 7)

## Regressions:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   bp ~
##     age               0.319    0.046    6.890    0.000
##     male              5.217    1.216    4.289    0.000
##   glu ~
##     age               0.240    0.039    6.123    0.000
##     male              3.695    1.029    3.590    0.000
```

Estimating the Model

```
partSummary(out, 8:10)

## Covariances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##   age ~~
##     male              0.000
##  .bp ~~
##    .glu              41.230    6.840    6.028    0.000
##
## Variances:
##                    Estimate  Std.Err  z-value  P(>|z|)
##    .bp              162.824   10.953   14.866    0.000
##    .glu             116.565    7.841   14.866    0.000
##     age             171.458   11.534   14.866    0.000
##     male              0.249    0.017   14.866    0.000
```

Visualizing the Fitted Model



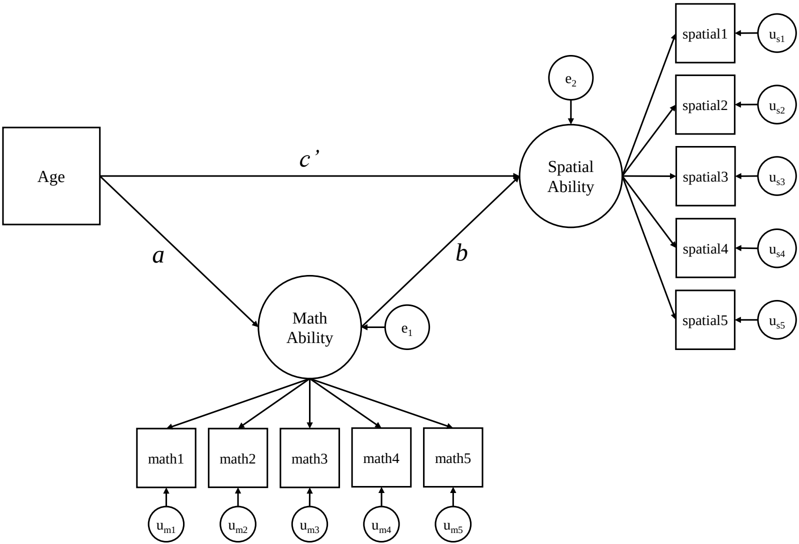
Teaser

Suppose we have the following theory about student performance.

- A students age affects their spatial reasoning ability.
- The effect of age on spatial reasoning ability is partially mediated by mathematical ability.
- We can measure spatial reasoning ability and mathematical ability with five-item tests.

We could translate this theory into the following *structural equation model*.

Teaser


Age
c'
a
b
Spatial Ability
e$_2$
spatial1
spatial2
spatial3
spatial4
spatial5
u$_{s1}$
u$_{s2}$
u$_{s3}$
u$_{s4}$
u$_{s5}$
Math Ability
e$_1$
math1
math2
math3
math4
math5
u$_{m1}$
u$_{m2}$
u$_{m3}$
u$_{m4}$
u$_{m5}$

References

Breiman, L. (2001). Statistical modeling: The two cultures (with comments and a rejoinder by the author). *Statistical Science*, *16*(3), 199–231.

Smaldino, P. E. (2017). Models are stupid, and we need more of them. *Computational Social Psychology*, 311–331.